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Adaptive Approaches in the Finite Element Method

by

Ivo Babuska
and
Werner C. Rheinboldt

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A Survey of A Posteriori Error Estimators and
Adaptive Approaches in the Finite Element Method¹⁾

by

Ivo Babuška²⁾ and Werner C. Rheinboldt³⁾

1. Introduction

In general, the objective of a computational solution of a practical problem is to obtain an acceptably accurate and reliable prediction of the behavior of the physical phenomena under study. For this the computation should produce not only an approximate numerical solution but also some reliable information about its accuracy.

The selection of the specific accuracy requirement depends strongly on the goal of the computation. Often it is desired to obtain detailed information about the solution itself. In other cases, the main focus is the value of a specified functional of the solution, as, for instance, a stress intensity factor in fracture mechanics, or a drag in fluid dynamics. Then, again in other cases, interest centers, say, on streamlines and/or on the existence of separation lines, as, for example, in seepage problems. Other goals may be the determination of certain critical data, as, collapse points or buckling points and their associated loads in structural mechanics, etc.

In connection with most of these goals we are interested in quantitative results which have a desired accuracy. For this the error has to be defined, that is, the exact results have to be specified with which the computed data are to be compared, a norm has to be prescribed under which the error is to be measured, and, last but not least, some procedure has to be established for estim-

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ating the desired error-data. On the other hand, for some of the indicated goals we may be interested only in qualitative behavior, as for instance the mentioned existence of separation lines. Such a requirement may lead into other branches of mathematics as, for example, into the qualitative theory of ordinary differential equations in the case of streamline problems.

In recent years, the use of the finite-element in engineering and science has become a multi-million dollar enterprise. Many programs exist and many more are being developed, (see eq. [1]). In line with the stated need for information about the accuracy of the computed solution we may expect that future programs will incorporate facilities for computing such error estimates. As indicated before, these estimates should correspond to the principal goals of the computation that can be selected by the user of the program, and the program should allow a choice of the norm under which the estimate is to be computed.

Such error-estimation facilities are certainly of direct importance in many applications, provided they can be guaranteed to be reliable. For instance, for the many types of certifications-computations required in the design of complex structures, nuclear plants etc., the availability of reliable estimates of the accuracy of the computed data is obviously very essential. In other cases, such estimates may reduce the total design effort and avoid unnecessary overdesign.

On the other hand, the availability of effective error estimates introduces the possibility of structuring the finite element computation so as to achieve a desired error tolerance at minimal cost or to provide the best possible solution within an allowable cost range. This is an important objective especially in view of the fact that in typical engineering applications an accuracy of, say, 2-5% is desirable, and any effort expended to obtain a much higher accuracy is a waste.

It is beginning to be a widely accepted observation that for realistic problems it is rarely feasible to design numerical processes which reliably and effectively achieve a desired accuracy at reasonable cost and yet which do not

utilize some form of adaptivity. At the same time adaptive approaches may also lead to a simplification of the input data needed for the program and hence free the user of part of the drudgery typical in preparing such input and of relieving lines of many of the a priori decisions required by most of today's programs.

While the need for adaptive approaches is well recognized, the precise meaning of the adaptation-concept remains vague and ill-defined. A classification of this concept may require close attention to results in such fields as automatic control theory, artificial intelligence, learning processes, etc. (see eq. [2], [3], [4], [5]). In our context, the availability of precisely defined reliable error estimators appears to be central to the design of effective adaptive procedures. The effectivity has to be measured here in terms of the size of the class of problems for which solutions within a specified range of accuracy are obtainable with minimal cost. Of course, the question of cost is here a very complex issue and still requires much thought.

By far the majority of todays finite element computations involve linear problems, although more and more attention is being paid to nonlinear phenomena. Not unexpectedly, there are many differences between linear and nonlinear problems, and the latter show many special features not present in the linear case. As stated at the outset, the objective of the computation should be a prediction of the behavior of the physical system under study. If we are concerned, for instance, with a mechanical structure, then the term "behavior" actually refers, say, to the deformations of the structure in response to the action of external influence quantities, such as loads, changes in material properties and geometrical data, etc. This means that the equations which govern the deformation of behavior variables also depend on a set of influence parameters and the objective is to assess the behavior of the solutions under general variations of the parameters. In the linear case, most of this variation is also assumed to be linear and we need only compute a few specific solutions to achieve our objective.

In nonlinear problems, however, the computation of a few specific solutions provides little or no insight into the behavior of the system and we are led to consider the set of all solutions depending on all the parameters in a specific range. In structural mechanics, this set is now usually called the equilibrium surface of the structure. The objective then becomes an analysis of the form and special features of this surface, as for example, the boundaries of the stability regions on the surface, etc. The principal tools for such an analysis are the continuation methods, as incremental processes, as they are called in structural mechanics. As before error estimations and adaptive approaches will have to be incorporated into these processes. This is as yet a wide-open research area.

This survey is intended to give an overview of some of the recent results that have become available about reliable, computable error estimators for finite element solutions as they exist in the design of adaptive solution procedures both in the linear as well as the nonlinear case. By necessity, the discussion had to be kept on a descriptive level and for all mathematical details we need to refer to the cited literature.

2. Accuracy Considerations in the Finite Element Method

The finite element approach to the solutions of suitable boundary value problems may be characterized loosely by the use of a weak bivariate form of the problem and the approximation of the solution by piecewise smooth functions. There is a growing number of introductions to the basic theory of the method; we refer only to [6], [7], [8], and [9] to mention a few.

Many of the available theoretical results concern an asymptotic analysis of the convergence rate of the method. Although this type of analysis is not too useful for assessing the accuracy of a specific solution, it provides important insight and offers guidelines for future efforts.

We illustrate this point with some observations from general practical experience with engineering problems. It is well accepted that in the case of fairly complex problems a relative accuracy of about 8-10% in the energy norm is achievable with about 1000 to 2000 degrees of freedom N . On the other hand, in order to obtain an accuracy of 3-5%, the rate of convergence should be at least of the order of N^{-1} . But, in the presence of singularities and for uniform meshes in two space dimensions the "standard method" provides often only a rate of $O(N^{-1})$ or worse, and hence the desired accuracy of 3-5% is practically infeasible. For three-dimensional problems the situation is even more difficult.

By the standard method we mean here the so-called h-version of the finite element method which is most widely used and studied. In that version, the degrees of the elements used for the computation are fixed and the increasing accuracy is achieved by the use of appropriate sequences of finer and finer meshes. The h-version indeed underlies the design of practically all of today's commercially available codes (see [1]).

Recently a new approach became available with the development of the so-called p-version and h-p-version of the finite element method. In the p-version, the meshes are fixed and the increasing accuracy is achieved by the use of elements with higher and higher degrees. The h-p-version combines the two approaches in a

judicious manner. It turns out that the architecture of a finite-element program based on the p-version or h-p-version will be significantly different from that of one of the standard programs founded on the h-version. Moreover, the mathematical properties of these different versions are rather distinct as well.

In order to illustrate the last point, we mention a typical rate-of-convergence result proved in [10], [11]. Suppose that the use of the h-version with a sequence of uniform meshes leads to a rate of convergence of order $N^{-\alpha}$ where α does not involve the (fixed) degree of the elements. There the p-version, in general, exhibits a rate of order $N^{-\beta}$ where

$$\alpha - \epsilon \leq \beta \leq 2\alpha - \epsilon$$

with any $\epsilon > 0$.

For the h-version the convergence rate is governed by the degree of the elements and the smoothness of the solution. More specifically, for a wide class of practical problems involving only singularities caused by corners of the domain, etc., it can be proved (see [12]) that there exist sequences of suitably refined (non-uniform) meshes for which the rate of the h-version is controlled solely by the degree of the elements. On the other hand, in [11] it was shown that in the same cases, the h-p-version provides for a convergence rate which is better than any polynomial in N or even exponential.

In general, the p-version is better in handling singularities (located at the vertices of elements) than the h-version does in the case of uniform meshes. Moreover, it has been shown in [13], [14], and [15] that the p-version is practically not influenced adversely by the presence of nearly incompressible materials. On the other hand, the h-version, especially when low order elements are used, has encountered difficulties due to this incompressibility. In many cases, the use of the so-called selectively reduced integration techniques helps to overcome these difficulties. These techniques are equivalent to some mixed method. For a typical analysis of problems of this type in one dimension we refer to [16].

These comments already indicate that future developments should take more account of the selection of the best form of the finite element method for particular classes of computational problems. In particular, the h-p-version appears to offer an attractive opportunity. It is certainly beginning to be recognized now that the use of high-order elements--when properly implemented in a hierachial manner--is more effective, in many cases, than the classical h-version approach. For some related references we refer to [13], [17] and references there. The p-version as well as the h-p-version were implemented in the commercially available program COMET-X for 2 and 3 dimensional problems. But it remains a research problem how to design effective adaptive approaches for the h-p-version.

3. A Posteriori Estimates and Adaptive Techniques.

The standard results about errors of finite element solutions concern almost exclusively a priori estimates. As noted before, as important as these estimates are for theory of the method, they are rarely useful in practice for estimating the error of a particular computed solution. For that we have to turn to a posteriori estimates which utilize information obtained during the solution process itself.

The use of a posteriori estimates means that, together with the approximate solution, the error estimator ϵ is computed which represents a measure of norm $\|e\|$ of the error e of that solution. As a measure of the reliability of the estimate we may introduce the effectivity index

$$\theta = \frac{\epsilon}{\|e\|} .$$

Clearly θ should be close to one when the error $\|e\|$ diminishes.

We distinguish estimators by the information about θ that can be guaranteed. An error estimator ϵ is an upper estimator, or a lower estimator if

$$\|e\| \leq C_U \epsilon$$

or

$$C_L \epsilon \leq \|e\| ,$$

respectively, where the constants depend on reasonably large class of possible solutions and on a (reasonably large) class of admissible finite element meshes, but not on the specific computed solution and its mesh. An upper or lower estimator is called "guaranteed" if $C_U=1$ or $C_L=1$, respectively. Finally, an estimator is asymptotically correct if

$$\lim_{\|e\| \rightarrow 0} \frac{\epsilon}{\|e\|} = 1$$

for a specified class of admissible meshes. In general, this class turns out to be more restrictive than that used in any property of ϵ to be an upper or lower estimator. For an analysis of these properties of various estimators

for a class of problems in one space dimension and for different norms, we refer to [18].

For practical application it is essential that $|\theta-1|$ is small, say of the order of 0.2 to 0.3 when the relative error is of the order of 10% and the order of 0.05 to 0.15 when it is in the range of 2-5%. In general, the size of $|\theta-1|$ is a much more important item of information than, say, the fact that $C_L=1$, that is, that ϵ is a guaranteed upper estimator. Theoretical results as well as experimental experience indicate that the type and method of construction of the finite element meshes is crucial for $|\theta-1|$ to be small. Not surprisingly, these classes of meshes can only be obtained by adaptive techniques.

The classical finite element method addresses positive definite, self-adjoint problems. Then the energy norm is the natural norm even though it is not always the most relevant one for practical applications. When classical forms of the finite element method are considered, the theory based on the energy norm is especially advantageous and, for instance, the well-known two-sided estimates of the error becomes available. But their practical computation is expensive, and, as all such global constructions of error estimators, they turn out to be unusable as a tool for adaptive mesh-constructions. In other words, we should look for error estimators ϵ with a local character. By this we mean that ϵ is composed of error indicators $n(\tau)$ each of which corresponds to a single element τ of the current mesh Δ and is computed from data about the solution on τ and at most the immediate neighbors of this element. The construction of the error-indicator $n(\tau)$ and their composition into the overall estimator ϵ depends on the particular norm. For example, for the energy norm one is led to the definition

$$\epsilon = \left[\sum_{\tau \in \Delta} n(\tau)^2 \right]^{\frac{1}{2}}.$$

All the estimators studied in [18] and [19] for the one-dimensional and in [20] for the two space dimensions have been local character.

The local nature of these estimates allows their use in the construction of meshes with optimal error behavior. There are various papers in the literature (see e.g. [21], [22], [23]) which address the characterization of optimal meshes for a prescribed number of nodal points and a specified norm. From a theoretical viewpoint these results are certainly important, since they provide information about the best possible error. But for practical applications the question of looking for an optimal mesh is not very effective. In fact, we are interested only in achieving an error that is nearly optimal and it turns out that there is considerable flexibility in the meshes for which this holds.

This and related results were presented and analyzed in depth in [19] for a class of one-dimensional problems. There it also shown that asymptotically (in a certain mathematical sense of the word) the meshes with optimal error are characterized by the equality of the error indicators $\eta(\tau)$ mentioned above. This represents the so-called equilibration principle for mesh-construction. It also turns out that this equilibration is essential to guarantee that θ leads to one (see [20]).

Although the results about the equilibration have been fully proved only for one-dimensional problems, all indications are that in some form or other they will hold in general. In any case, we should not expend too much costly effort in the determination of an optimal mesh, but instead construct a succession of meshes which are as much as possible equilibrated and, hence, provide for solutions with nearly optimal errors for the corresponding numbers of degrees of freedom.

The algorithms for the adaptive construction of the desired sequence of nearly equilibrated meshes presently are still largely of a heuristic nature, (see e.g. [3], [24], [25] for the beginnings of a theory of such algorithms.) In principle, at each stage of the process any element τ of the current mesh Δ is refined for which the error indicates $\eta(\tau)$ exceeds some threshold and its refinement is predicted to be within the expected equilibrations range of the

resulting mesh. Since a new solution has to be computed on the refined mesh which is expensive, there is a need for balancing the quality of the meshes with the effort of designing them. In fact, practical interest centers not only on near optimal errors for the number of degrees of freedom used, but for the total machine time expended. This corresponds also to the stopping criterion which should terminate the refinement process if either the required accuracy has been achieved, or the machine-time and storage-estimations are exceeded. This is another reason in favor of working with sequences of nearly equilibrated meshes, since then at each stage the error of the computed solution is expected to be nearly optimal for that stage.

4. An Adaptive Finite Element System

Although the mathematical theory must be the basis of understanding the concepts discussed in the previous sections, it is essential to test them experimentally on a meaningful set of realistic problems. In order to facilitate such an experimental program a prototype system incorporating these ideas was developed at the University of Maryland in recent years. It was dubbed FEARS, short for Finite Element Adaptive Research Solver. Details about the system may be found in [24], [25], [26], [27] and [28].

FEARS was never intended to become a production system. Instead, in its design we were guided by the following axioms:

- (1) To verify that the general concepts outlined earlier can be effectively implemented in a robust system.
- (2) To study the influence of the adaptive environment upon finite-element program architectures, the relevant data structures, etc.
- (3) To assess some potentialities for procedural parallelity in the method.

FEARS is designed to solve a class of elliptic systems of two partial differential equations in two dimensions. More specifically, with the notation

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \left(\frac{\partial u_1}{\partial x_1}, \frac{\partial u_2}{\partial x_1}, \frac{\partial u_1}{\partial x_2}, \frac{\partial u_2}{\partial x_2} \right)^T,$$

and similarly for $\mathbf{v} = (v_1, v_2)^T$, the admissible problems are assumed to be written in the weak form

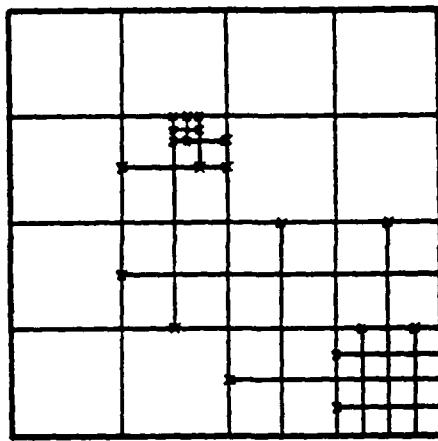
$$\begin{aligned} B(\mathbf{u}, \mathbf{v}) &= \iint_{\Omega} \left[\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right)^T \mathbf{A} \frac{\partial \mathbf{v}}{\partial \mathbf{x}} + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right)^T \mathbf{B} \mathbf{v} + \mathbf{u}^T \mathbf{B}^T \frac{\partial \mathbf{v}}{\partial \mathbf{x}} + \mathbf{u}^T \mathbf{C} \mathbf{v} \right] d\mathbf{x}_1 d\mathbf{x}_2 \\ &+ \int_{\Gamma} \mathbf{u}^T \mathbf{A}_0 \mathbf{v} ds = \iint_{\Omega} (D \frac{\partial \mathbf{v}}{\partial \mathbf{x}} + \mathbf{E} \mathbf{v}) d\mathbf{x}_1 d\mathbf{x}_2 + \int_{\Gamma} \mathbf{e}^T \mathbf{v} ds \end{aligned}$$

Here A, B, C, A_0 are piecewise constant matrices while the matrices D, E and the vector e are given functions of the independent variables x_1, x_2 . The matrices are supposed to be such that the problem is elliptic and the finite element procedure leads to symmetric linear equations.

The domain $\bar{\Omega} \subset \mathbb{R}^2$ is assumed to be the union of a relatively small number of curvilinear quadrilaterals, called 2D subdomains. Here we were influenced by the successful substructuring concept in engineering. The number of 2D subdomains is limited by storage consideration; in practice, we did not exceed the number 20. The (relatively open) sides of the quadrilaterals are called 1D-subdomains and the vertices 0D-subdomains. The 1D-subdomains are assumed to be circular arcs, each characterized by its vertices and curvature. The union of all 2D-, 1D- and 0D-subdomains is the set $\Gamma \subset \Omega$ used in the definition of the weak form $B(u,v)$.

The matrices A, B, C are constant on each 2D-subdomain while A_0 is defined on Γ and constant on each 1D-subdomain. Natural boundary conditions may be introduced via the vector e .

Since each 2D-subdomain is assumed to be diffeomorphic to the unit square in \mathbb{R}^2 , the finite element meshes on the 2D-subdomains are defined first on the unit square and then mapped over into the corresponding curvilinear quadrilateral. A typical mesh on the square is shown in Figure 1, it consists of conforming bilinear elements of different size. The vertices marked by crosses are irregular nodal points. The values of the solution in these points are determined by the conformity. The mapping into the quadrilateral is constructed by blending techniques from the formulas of the circular side-segments of that quadrilateral.

Figure 1

The admissible error norms in FEARS have the form

$$\|e\| = \left\{ \iint_{\Omega} \left[\left(\frac{\partial e}{\partial x} \right)^T Q \frac{\partial e}{\partial x} \right]^{p/2} dx_1 dx_2 \right\}^{1/p}$$

where Q is an input matrix and e the difference between the exact and the finite element solution. The choice of Q may be different on each 2D-subdomain. For the energy norm in absence of B and C we use $Q = A$.

The error indicators $\eta(\tau)$ are computed for each element of the mesh Δ of Ω as functions of the following quantities:

- (1) The size of the element τ ,
- (2) The jump of the derivatives across the boundary of τ ;
- (3) The projection of the residual of the computed solution on τ onto the space of constant functions on τ .

As mentioned in the previous section the error estimate ε of the solution is then composed of the individual $\eta(\tau)$. For further details about the error estimation procedure and its theory we refer to [20].

In order to accomodate the type of meshes and the refinement process a special data structure was designed in the form of a rooted tree. A detailed

description of this design and the corresponding access algorithms is given in [28]. The data structure provides for a natural pivot ordering based on nested dissection for the solution of the global stiffness matrices by LU-decomposition.

The adaptive mesh-refinement algorithm follows the general outline given in the previous section. An element in the base-mesh on the unit square is refined by dividing it into four congruent subsquares. The threshold used in deciding which elements are to be subdivided is chosen to be the predicted largest error indicator value in the next mesh. This prediction is based on a simple extrapolation of the change of the error indicators from the previous to the current mesh. The threshold may be revised upward if the resulting storage or machine-time requirements are predicted to exceed the available resources, or the desired accuracy is expected to be reached.

In order to cut down on the cost of solving too many large linear systems, FEARS employs two different types of refinement steps, called long and short passes, respectively. A long pass involves a recomputation of the full solution on the entire mesh while in the short passes only certain local computations are carried out which allow for a decision about further refinements. The short passes avoid the necessity of computing a solution on a mesh which differs only in a few elements from the previous one. The combination of short and long passes is controlled again adaptively.

5. Some Sample Computations with FEARS

During the past year FEARS has been applied to an extensive range of practical problems and test problems. In the selection of the test-problems, we were guided by the desire to establish a worthwhile set of benchmark problems. Such a set should include not only problems for which the assumptions of the underlying, available mathematical theories are valid, but also some where they fail.

In assessing the outcome of any particular computational experiment, we used a number of criteria, including, in particular the following two:

- (1) Computation or estimation of the effectivity index θ as a function of the relative accuracy of the solution for the given norm.
- (2) Calculation of the observed rate of convergence as a function of the number of degrees of freedom and comparison with the theoretically expected rate for the particular type of problem.

It is impossible here to give more than a few typical examples of our experimentation with FEARS. Those included here were clearer mainly as an illustration of the capabilities of the system rather than in support of any specific point about the concepts discussed here.

Example 1: Consider the planar elasticity problem for the ring shown in Figure 2

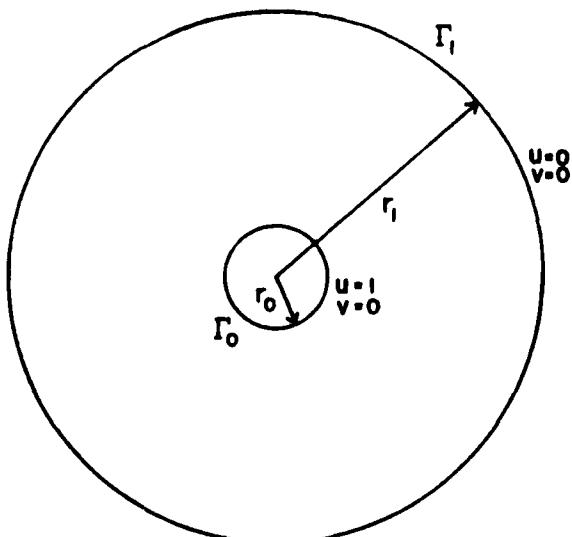


Figure 2

with prescribed displacements

$$u = 1, v = 0 \quad \text{on } \Gamma_0$$

$$u = v = 0 \quad \text{on } \Gamma_1$$

on its boundary lines. The material is assumed to be isotropic and homogeneous with modules of elasticity $E = 1$ and variable Poisson ratio ν . The exact solution of this plane-strain problem is known analytically. The error of the approximate solution in the energy norms is given by

$$\|e\|_E = (W_{FE} - W_{EX})^{\frac{1}{2}}$$

where W_{EX} and W_{FE} denote the energy of the exact and the finite element solution, respectively.

Since the problem has various symmetries, only a quarter of the ring needs to be used. In FEARS this domain is taken as one subdomain. The following Table 1 exhibits some experimental data for this problem and the following cases

$$r_1 = 1, r_0 = .5, \text{ and } r_0 = .1$$

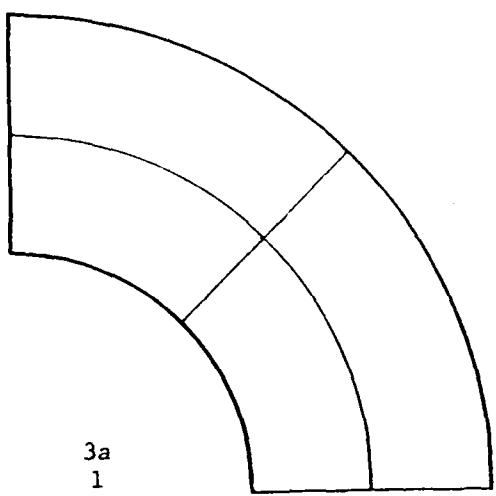
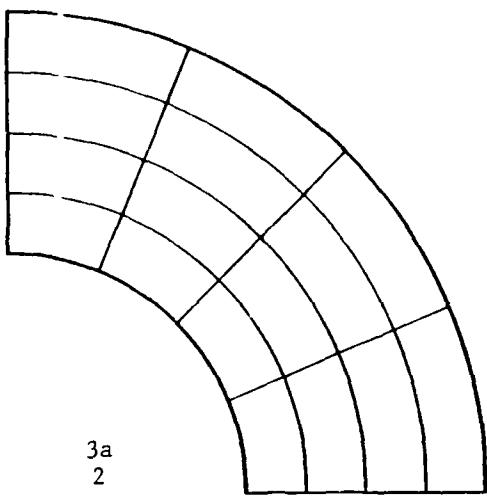
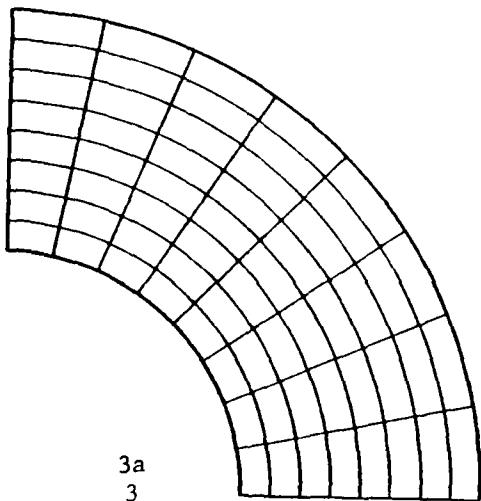
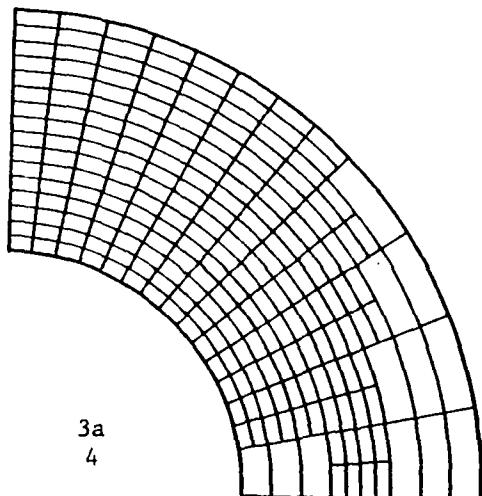
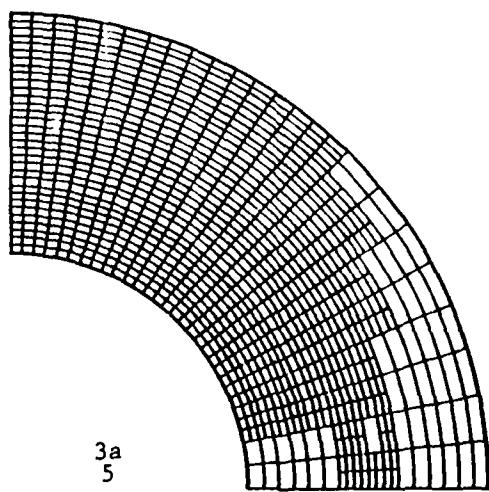
$$\nu = 0.3, \nu = 0.49, \text{ and } \nu = 0.499$$

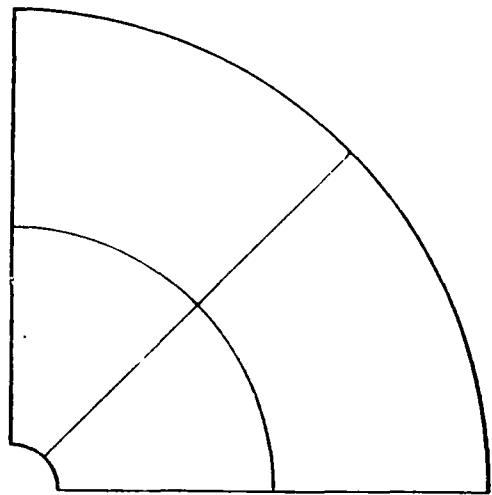
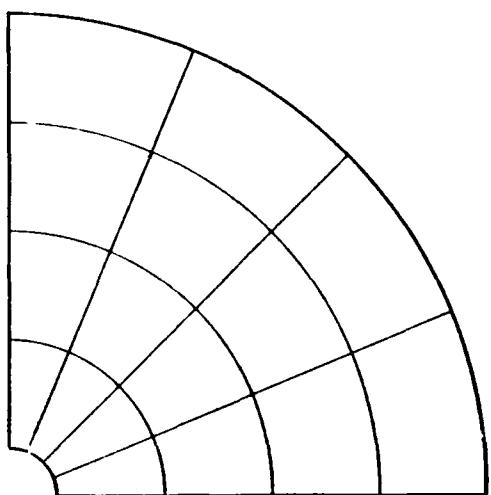
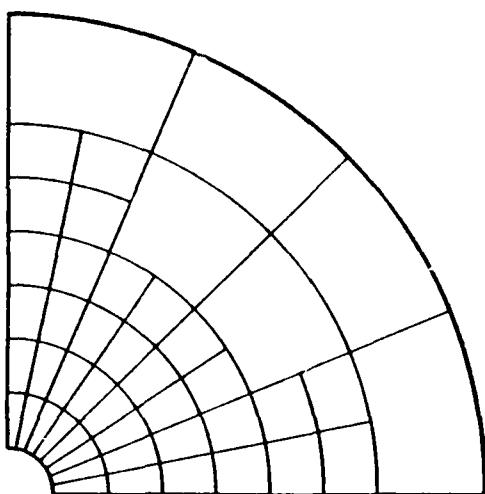
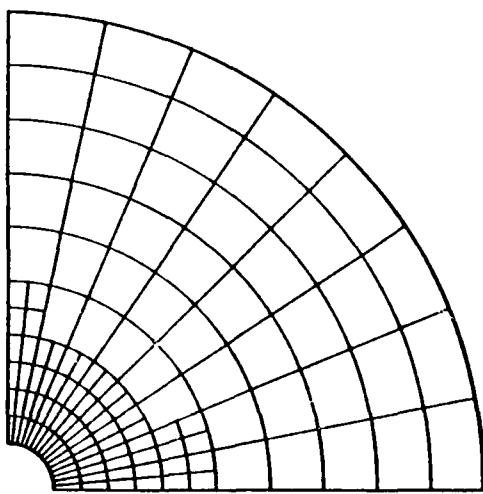
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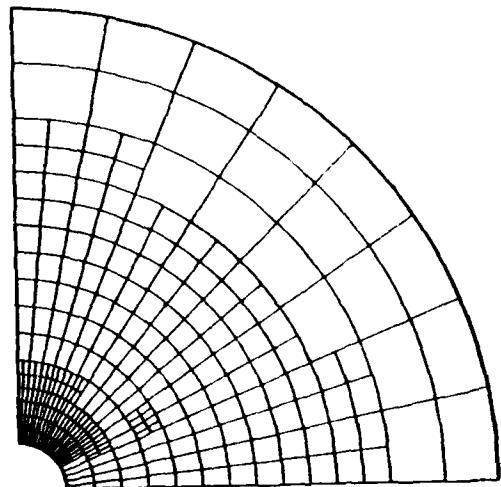
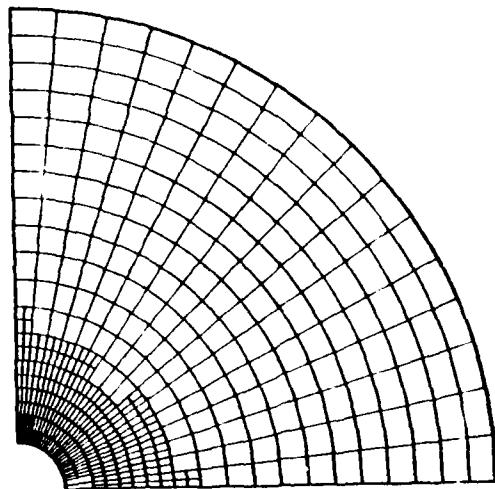
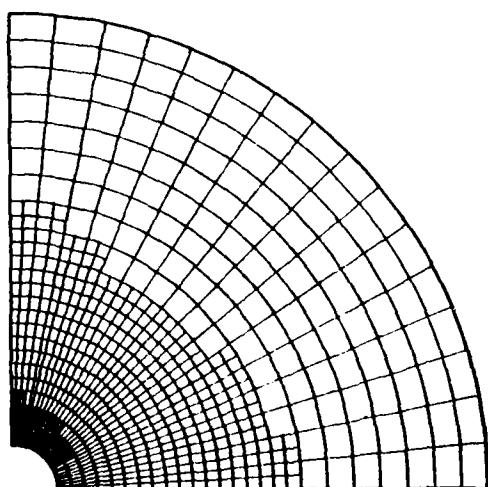
- (1) The inside radius r_0
- (2) The Poisson ratio ν
- (3) The number of elements
- (4) The number of degrees of freedom
- (5) The relative error in percent of the exact solution: $100 \|e\|_E / \|u_{EX}\|_E$
- (6) The computed relative error $100 \epsilon / \|u_{FE}\|_E$
- (7) The effectivity index θ

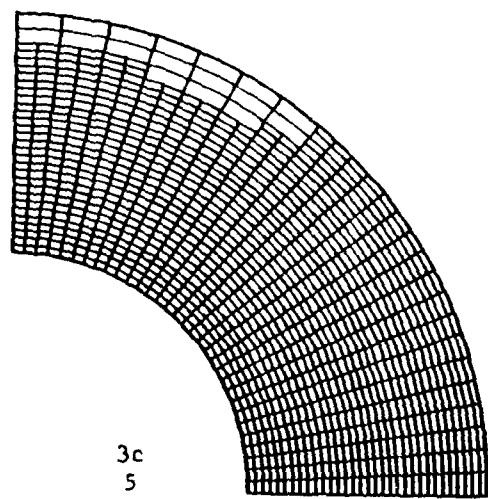
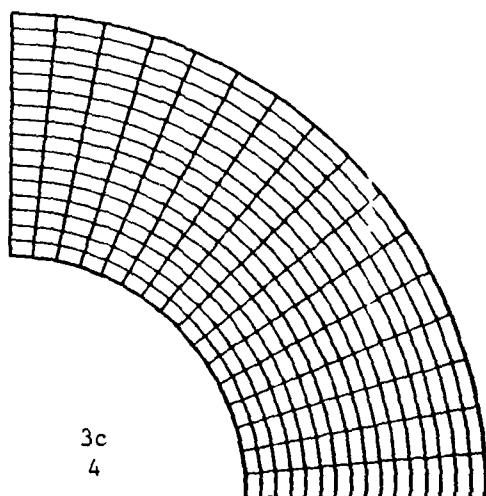
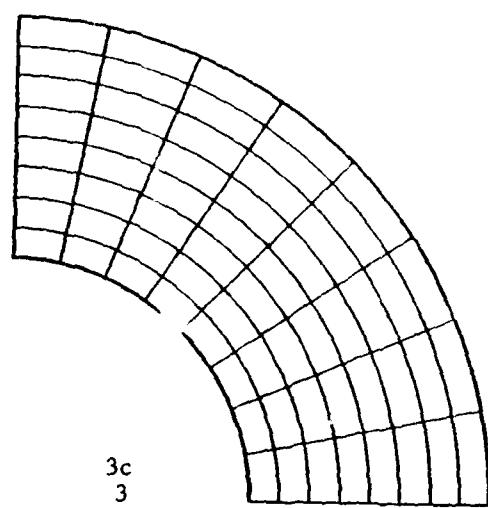
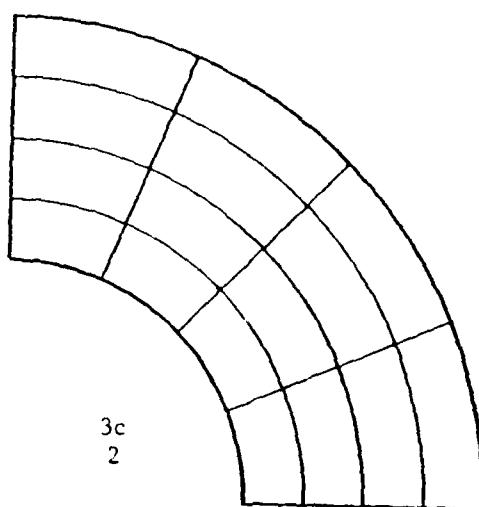
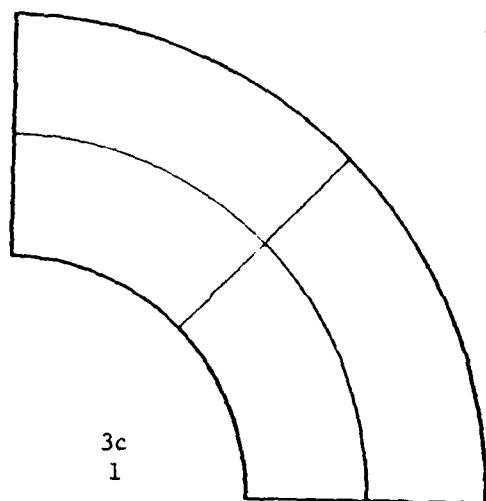
(1)	(2)	(3)	(4)	(5)	(6)	(7)
Fig. 3a 0.5	0.3	4	4	16.15	17.67	1.108
		16	24	8.47	8.42	0.997
		64	112	4.33	4.28	0.990
		220	402	2.36	2.40	1.017
		877	1674	1.17	1.19	1.015
Fig. 3b 0.1	0.3	4	4	52.42	40.44	0.871
		16	24	31.40	21.99	0.734
		46	74	18.81	15.20	0.822
		121	708	11.18	9.99	0.899
		280	493	7.40	6.97	0.979
		481	873	5.63	5.37	0.956
		820	1520	4.13	4.03	0.977
Fig. 3c 0.5	0.49	4	4	78.65	72.95	1.180
		16	24	42.15	39.39	1.014
		64	112	21.74	20.83	0.980
		256	480	10.98	10.78	0.988
		973	1878	5.69	5.68	1.000
Fig. 3d 0.1	0.49	4	4	169.66	107.88	1.252
		16	24	97.89	67.50	0.965
		64	112	58.11	39.59	0.788
		226	418	34.57	24.94	0.764
		436	791	23.56	20.86	0.909
Fig. 3e 0.5	0.499	4	4	261.32	111.33	1.192
		16	24	141.78	83.28	1.019
		64	114	73.91	58.24	0.979
		250	467	37.85	35.07	0.990
		820	1542	21.85	21.38	1.002
Fig. 3f 0.1	0.499	4	4	523.43	129.12	1.314
		16	24	290.85	98.12	1.037
		64	114	162.97	76.33	0.895
		253	473	92.69	57.00	0.839
		376	681	74.65	55.30	0.924
		694	1276	54.87	46.90	0.974

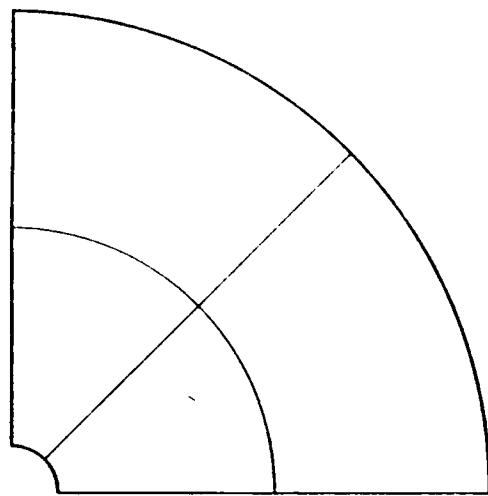
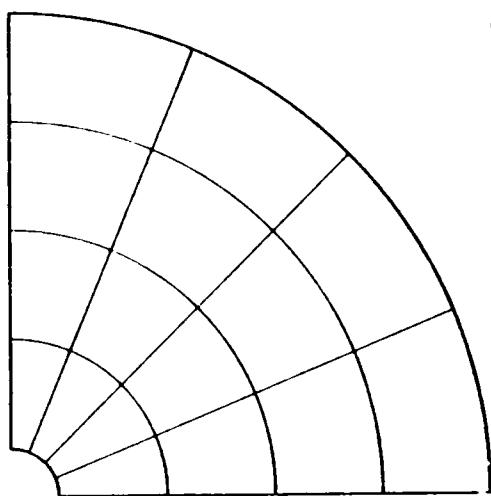
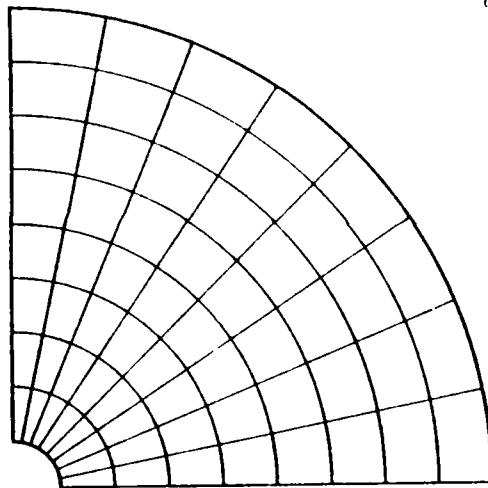
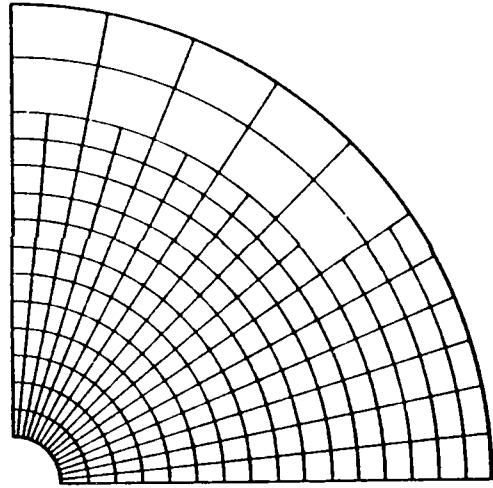
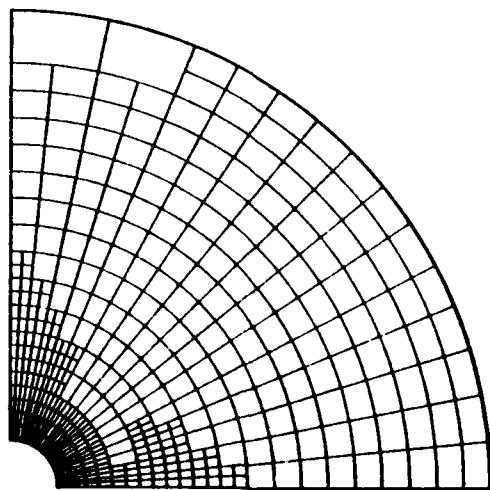
Table 1.

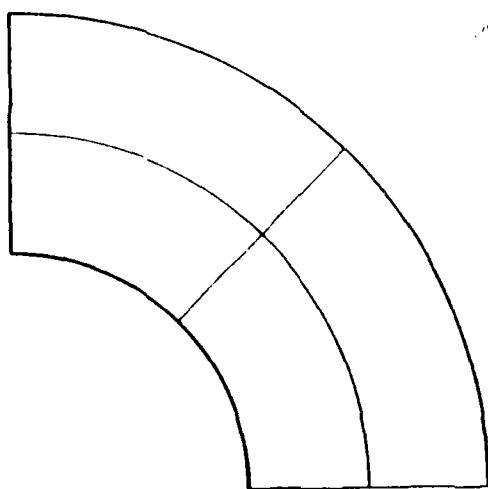
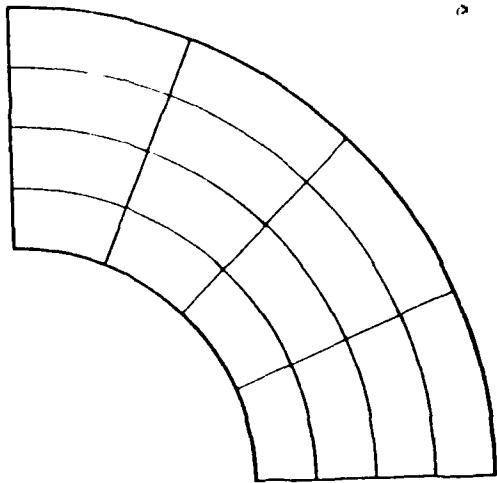
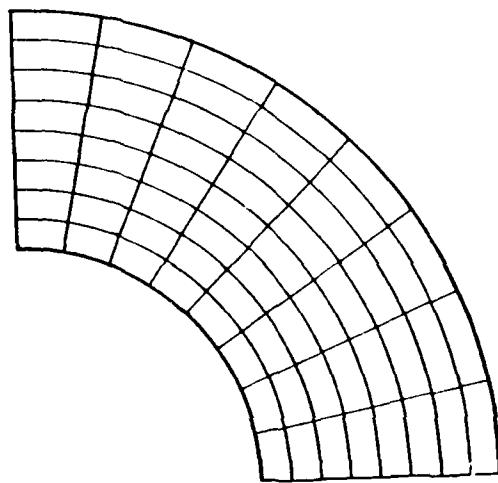
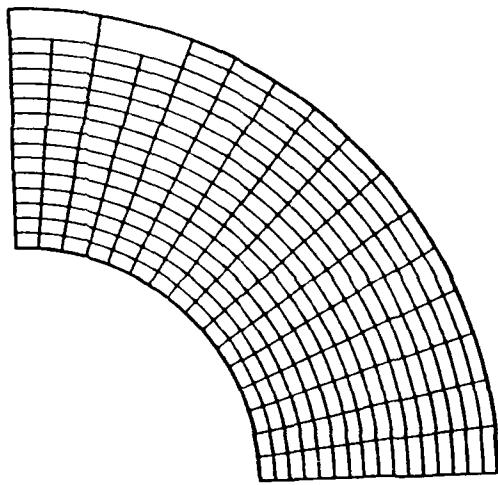
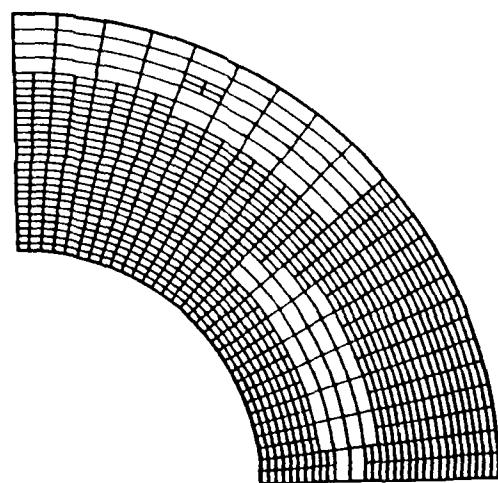
3a
13a
23a
33a
43a
5

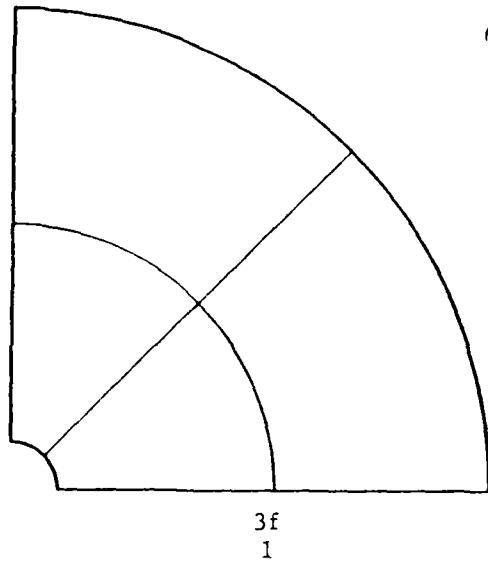
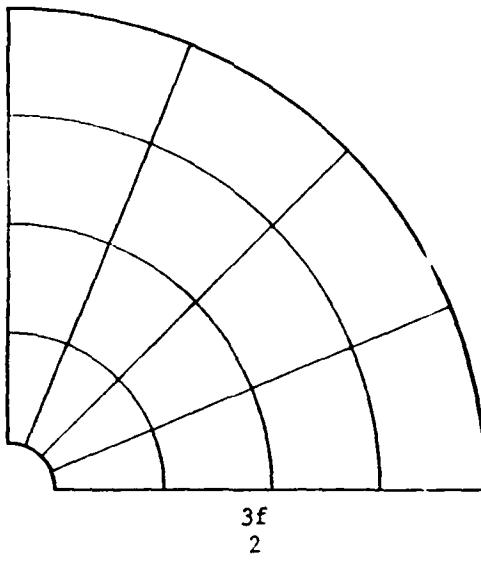
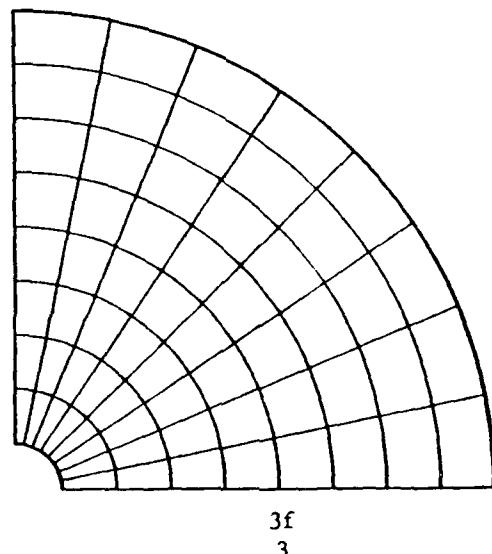
3b
13b
23b
33b
4

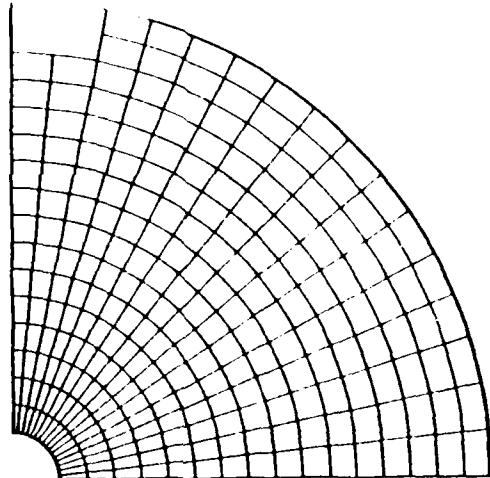
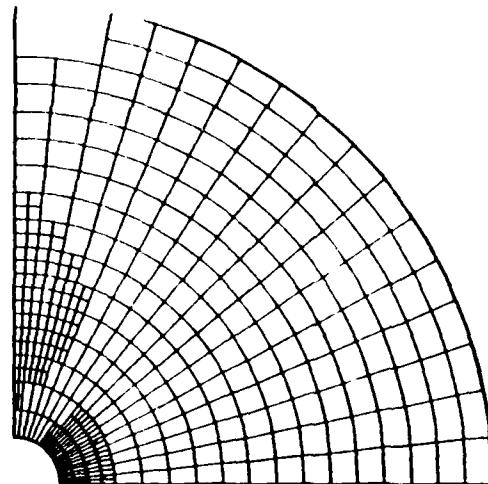
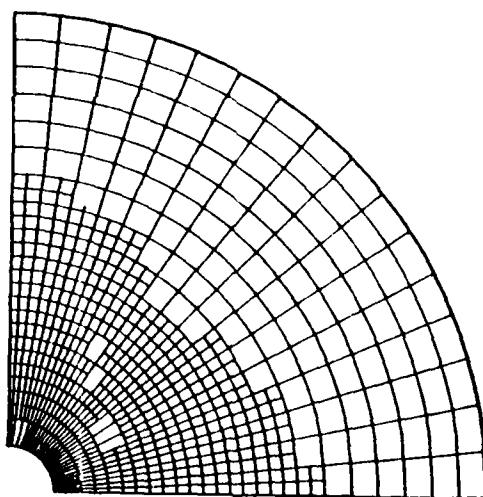
3b
53b
63b
7



3d
13d
23d
33d
43d
5

3e
13e
23e
33e
43e
5

3f
13f
23f
3

3f
43f
53f
6

Figures 3a-3f show sequences of meshes adaptively constructed by FEARS in these cases. The cases are denoted as follows:

Fig.	r_0	ν
3a	.5	.3
3b	.1	.3
3c	.5	.49
3d	.1	.49
3e	.5	.499
3f	.1	.499

The table shows clearly the high quality of the error estimators independently of the value of ν . Clearly $|\theta - 1|$ satisfies the earlier stated requirements. The table also indicates the increasing ineffectivity of the h-version when $\nu \rightarrow 0.5$, that is, where the material becomes more and more incompressible. Figure 3 shows that the adaptive procedure refines the meshes in a very plausible manner. This point will become more evident in the next example. The meshes also show certain small anomalies. Due to the threshold-algorithm mentioned in the previous section, elements someplace miss barely in being refined.

Example 2: Consider the plane-strain problem of a cracked panel as shown in Figure 4. The material is again assumed to be isotropic and homogeneous case $E=1$ and, plane strain, and Poisson ratio $\nu = 0.3$. Again because of symmetry we may work only with the upper right quadrant.

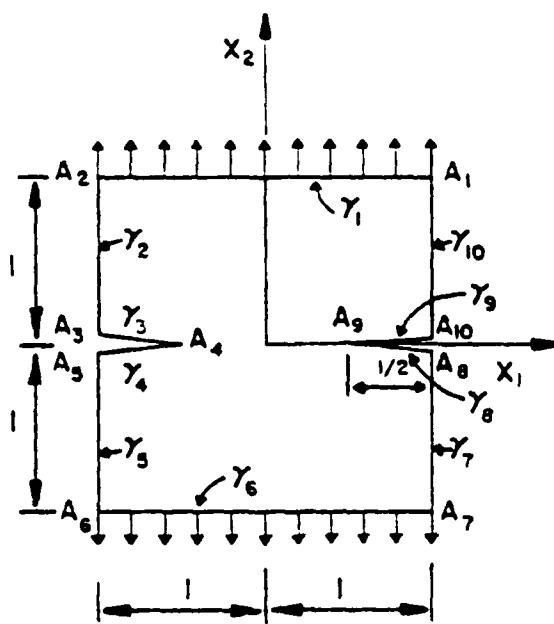


Figure 4

Clearly at the tip of the cracks, i.e. at A_4 and A_9 , the solution is singular.

In general, for smooth solutions, that is, without the cracks, the rate of convergence of the h-version cannot exceed $O(N^{-\frac{1}{2}})$. With the irregularity and for uniform meshes it reduces to $O(N^{-\frac{1}{2}})$ independently of the degree of the elements.

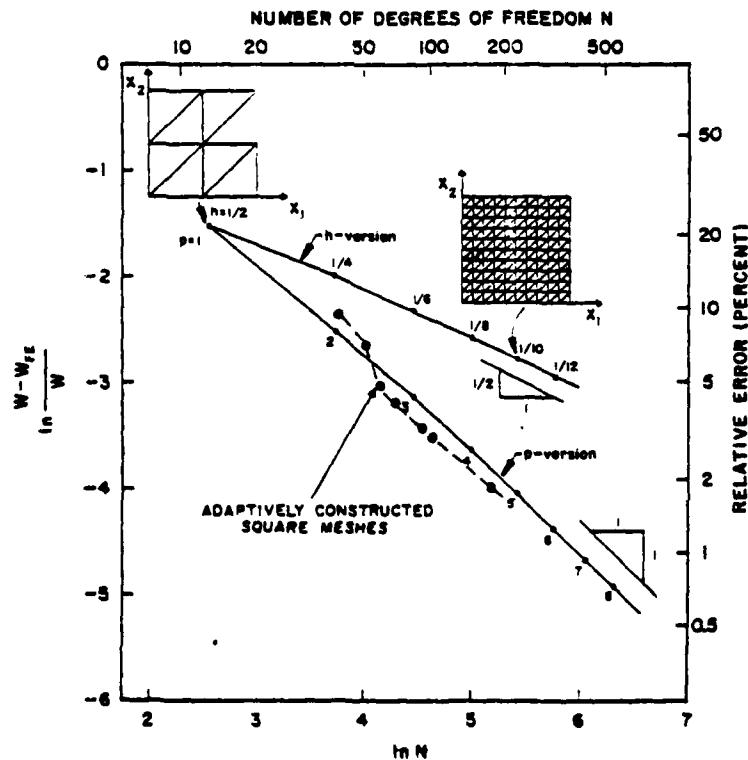
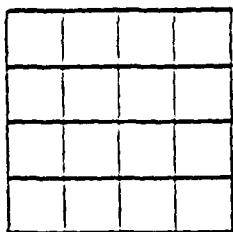
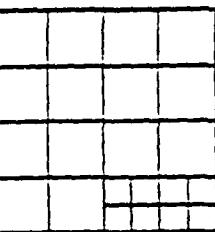


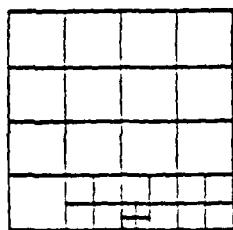
Figure 5



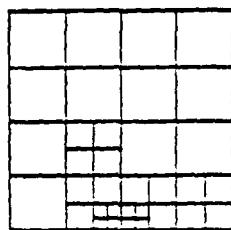
28



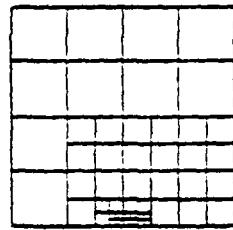
N = 54



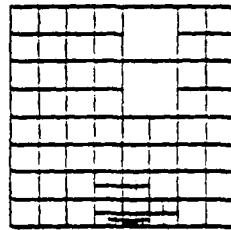
N = 63



N = 72



N = 101



N = 180

Figure 6

Figure 5 shows the accuracy of the solution in terms of the error in the energy (not the energy norm) for the adaptively constructed meshes given in Figure 6. In addition, Figure 5 includes a comparison of the performance of the h-version and the p-version. More specifically, for a partition of the domain into eight congruent triangles the p-version clearly exhibits twice the rate of convergence as the h-version for uniform meshes. This agrees with the general result mentioned in Section 2. We also observe the earlier noted best possible rate of order $N^{-\frac{1}{2}}$ for the sequence of adaptively constructed meshes. Of course, the corresponding energy rate is $O(N^{-1})$.

In problems of this type the stress-intensity factor is an important quantity for engineering considerations. It represents a partial derivative with respect to the crack length. Accordingly, an error estimator of the stress intensity factor can be obtained from a derivative of the error estimators of the solutions. It can be shown that under suitable conditions and for the sequence of adaptively constructed meshes these error estimators of the stress intensity factor are asymptotically correct.

In order to test this, the boundary conditions of the present problem were adjusted by a proper boundary condition on the sides A_1A_2 and A_6A_7 so that the solution is known. Table 2 shows some results computed with FEARS for this case.

N	Solution error		Error of stress-intensity factor	
	rel. error	ϵ	rel. error	ϵ
22	21.4%	0.59	12.0%	0.38
52	12.6%	0.74	4.3%	0.60
109	7.6%	0.85	1.7%	0.76
238	4.7%	0.93	0.6%	0.95

Table 2.

As the table indicates, the rate of convergence of the stress intensity factor is twice as high as that of the solution. Under appropriate conditions this can be proved theoretically. We also note that the effectivity index of the stress intensity factor is approximately the square of that for the solution.

Example 3: As a final example we consider a seepage problem involving a piezometric head u which satisfies the equation

$$\sum_{i=1}^2 \frac{\partial}{\partial x_i} a \frac{\partial u}{\partial x_i} = 0$$

where a represent the permeability and

$$v_i = -a \frac{\partial u}{\partial x_i}, \quad i=1,2$$

are the velocity components. Figure 7 shows the domain partitioned into sixteen 2D-subdomains. In each one of these subdomains the permeability is assumed to be constant. The values of the constants are specified in Table 3. As is typical for problems of this kind, we set $u = x_2$ at the upper boundary and use $\frac{\partial u}{\partial n} = 0$ along the vertical and lower sides of the domain.

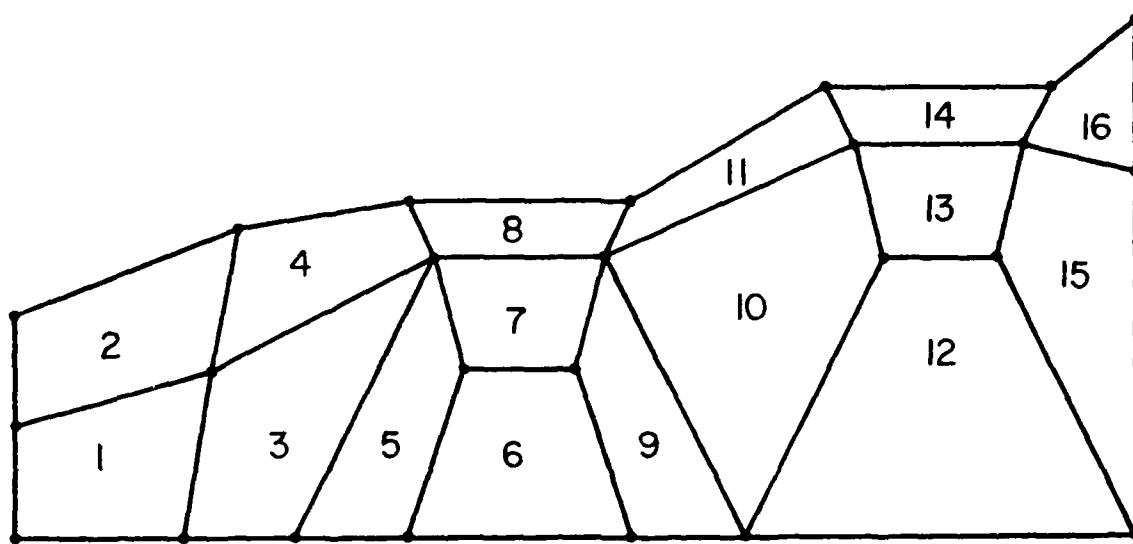


Figure 7

sub domain	a	sub domain	a
1	5.0	9	1.0
2	5.0	10	5.0
3	5.0	11	5.0
4	5.0	12	5.0
5	1.0	13	60.0
6	2.0	14	1000.0
7	70.0	15	5.0
8	1000.0	16	5.0

Table 3.

For the error assessment the following two norms were used

(1) The energy norm

$$\|u\|_E = \left[\int_{\Omega} \left(\sum_{i=1}^2 \left(\frac{\partial u}{\partial x_i} \right)^2 dx_1 dx_2 \right)^{1/2} \right]$$

(2) The velocity norm

$$\|u\|_V = \left(\int_{\Omega} \left(\sum_{i=1}^2 v_i^2 dx_1 dx_2 \right)^{1/2} \right)$$

Table 4 gives some experimental data obtained with FEARS for this problem.

(The effectivity index θ is only available for the energy norm). (The exact energy was computed by still larger computations and various extrapolation techniques, so that the true error can be established.)

Norm	No. of elements	Energy	$100 \ \epsilon\ / \ u_{FE}\ $	θ
$\ \cdot\ _E$	451	6.7867	8.16	1.012
	850	6.7706	6.92	0.982
$\ \cdot\ _V$	478	6.8035	7.73	N.A
	886	6.7776	6.16	N.A.

Table 4.

The adaptive approach based on the energy norm is expected to perform better than that based on the velocity norm when energy norm of the errors is considered. This agrees with the data in Table 4. The corresponding number of elements in the sixteen subdivisions is given in Table 5.

sub domain	Energy norm		Velocity norm	
1	19	28	40	46
2	55	100	55	79
3	16	37	16	28
4	10	19	34	34
5	10	25	4	22
6	10	34	4	70
7	4	4	7	25
8	7	7	31	52
9	31	97	22	25
10	76	115	55	100
11	55	91	40	58
12	64	103	40	130
13	13	19	31	55
14	4	19	31	73
15	19	28	16	19
16	58	124	52	70
Total N	451	850	478	886

Table 5

We end this section with a few comments about the overall performance of FEARS. The system includes a set of timers and counters to measure the cost of various program-modules. As an example, Table 6 gives some relative times for the seepage problem of example 3 in the case of the mesh with 850 constructed adapted adaptivity using the energy norm.

Module	
Refinement and related tasks	2.74%
Elimination of the blocks corresponding to the 2D subdomains and overall back substitution	51.81%
Elimination of the blocks corresponding to the 1D-and 0D- subdomains	26.14%
Computation of the error indicators	19.31%
Total	100.00%

Table 6

The separation of the elimination process into a two-dimensional and lower-dimensional level corresponds to the decomposition algorithm based on the earlier mentioned tree-structure for the data.

The table provides, of course, only some information about the relative machine time for the final mesh of the process. Experience has shown that the solution on the final mesh may be expected to take about 50% of the total machine time for the overall procedure. Moreover, in our runs with FEARS we found that the time of the solution on the final mesh is order N^λ , $1.5 \leq \lambda \leq 1.9$, where N is the final number of degrees of freedom. If one takes into account that the final mesh provides a considerably higher accuracy than any a-priori constructed mesh with the same number of degrees of freedom N , then the overall efficiency of the process becomes readily apparent.

6. A Convection-Diffusion Problem

So far essentially all our discussions concerned self-adjoint linear problems. For the remainder of this survey we turn now to problems of a different nature.

Convection-diffusion problems are important in many applications. In one dimension a typical form of such problems is as follows:

$$\epsilon u'' + a(x) u' + b(x) u = f(x), \quad 0 < x < 1$$

$$u(0) = \alpha, \quad u(1) = \beta$$

It is well-known that the solution shows boundary layer effects when ϵ is small either because of the boundary conditions, non-smoothness of f , or changing sign of a . Hence it is not surprising that the numerical solution of such problems is not easy and runs into typical difficulties for small ϵ . In the design of solution procedures a major concern is its robustness and stability with respect to ϵ , that is, a uniformity of performance for all ϵ in the interval $0 < \epsilon < 1$. In addition, in the application of the finite element method special care must be taken in the selection of the trial and test functions.

Because of their boundary layer character problems of this type use some form of the L_1 norm which is the best in various applications. More specifically, in connection with the finite element method it becomes necessary, for theoretical reasons, to introduce the following mesh dependent $L_{1,\Delta}$ -norm:

$$\|u\|_{L_{1,\Delta}} = \int_0^1 |u| dx + \sum_{i=1}^N \frac{h}{2}(h_i + h_{i+1})|u(x_i)|$$

where $x_0 = 0 < x_1 < \dots < x_N < x_{N+1} = 1$ are the nodal points and $h_j = x_j - x_{j-1}$, $j = 1, \dots, N+1$, the mesh-steps. For details about these norms we refer to [29], [30]. In [31] it was shown for the case of piecewise linear

test functions that the desired uniformity of performance with respect to ϵ requires the use of trial functions which depend on ϵ , the values of the coefficient functions a, b in the elements and their sizes.

The use of these functions can be combined with a posteriori error estimations under $L_{1,\Delta}$ -norms and the corresponding adaptive approaches. Once more it can be shown that these estimators are asymptotically correct. For details we refer to [32] and further forthcoming papers.

As an illustration consider the problem

$$-\epsilon u'' + v(x) u'(x) = 0, \quad 0 < x < 1$$

$$u(0) = 2\tanh\left(-\frac{1}{\epsilon}\right), \quad u(1) = 0,$$

with

$$v(x) = 2\tanh\left(\frac{x-1}{\epsilon}\right), \quad 0 \leq x \leq 1$$

This is a model of Burger's equation

$$-\epsilon u'' + uu' = 0$$

with the exact solution $u = v$. Figure 8 shows the accuracy of the computed

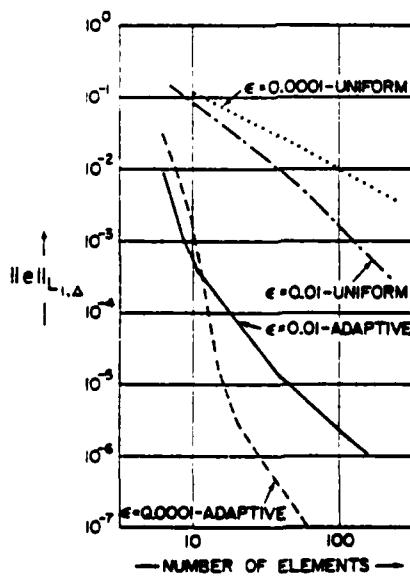


Figure 8

solutions for uniform meshes as well as adaptively constructed meshes for various values of ϵ . In all cases the $L_{1,\Delta}$ -norm was used. The drastic difference between the performance in the two cases is certainly very evident.

Figure 9 presents the so-called grading function $\xi(x)$ of the final, adaptively constructed mesh (see eq. [18], [32]).

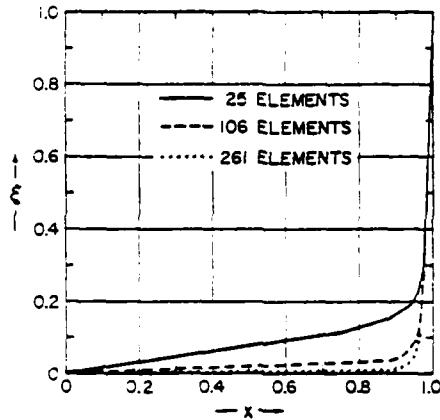


Figure 9

The relation between this mesh and the function ξ is given by

$$\xi(x_i) = \frac{i}{N}$$

where $\{x_i\}$ again denotes the nodal points. The refinement in the neighborhood of $x = 1$ where the boundary layer occurs is clearly visible. The quality of the adaptively constructed mesh sequence is apparent from Figure 9 where the

observed rate of convergence is $O(N^{-2})$ which is the maximally possible rate for linear elements.

Finally Figure 10 exhibits the behavior of the effectivity index of the estimators for both the sequence of uniform meshes and the adaptively constructed ones.

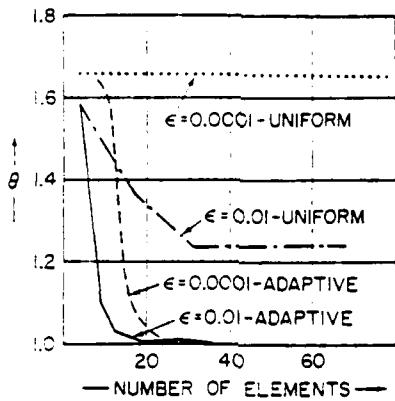


Figure 10

Again the high quality of the estimators as well as that of the adaptive mesh sequence is certainly evident. But even for the uniform meshes the estimators are not too bad. Moreover, the figure illustrates the theoretically supported fact that the estimator is simultaneously an upper and a lower one.

The general layout used for the program in this case is similar to that of FEARS; in particular the adaptive procedure corresponds entirely to that employed here.

Although the model problem is only one-dimensional, the ideas formulated here are not restricted to that case and we expect that they can be applied also to more general situations in higher dimensions.

7. A Parabolic Problem

Among the most effective methods for solving parabolic problems, as, for instance, the transient diffusion problem

$$\frac{\partial u}{\partial t} = \Delta u + f ,$$

is the method of lines. Its basic idea is to discretize the equation in the space variables and to solve the initial value problem for the resulting system of ordinary differential equations by one of today's sophisticated adaptive ODE-solvers.

If the finite element method is used for the space discretization, then the concepts discussed in the earlier sections suggest the following tasks:

- (1) Develop strategies for the adjustment of the tolerance in the ODE-solver so that the errors caused by the discretizations in space and time are comparable.
- (2) Design and analyze error estimators for the total error.
- (3) Develop algorithms for combined space and time discretization.

Two approaches to the third task have been considered. The first one consists in moving the nodal points of the finite element mesh in line with information obtained during the solution of the ordinary differential equations. This was discussed in [33], [34], [35]. In the second approach the mesh remains fixed during certain intervals in time. The length of these intervals is chosen adaptively and at the transition between them the mesh is changed discontinuously, for instance, by refining (or de-refining) it as in FEARS. For further details we refer to [36], [37].

The adaptive control, of course, requires the availability of an appropriate theory of a posteriori error estimates. Such a theory was introduced in [35], [37], [38]. It leads to estimators which turn out to be upper and lower estimators that are asymptotically correct under suitable conditions.

A general procedure for a class of parabolic equations in one space variable has been implemented based on the mentioned error estimators and the second of the above approaches for adapting the finite element mesh. As an illustration we present some computational results for the problem

$$\frac{\partial u}{\partial t} = \frac{d}{dx} a(x) \frac{du}{dx} - b(x)u + f(t,x), \quad 0 < x < 1, \quad t > 0$$

$$u(t,x) = g(t,x), \quad x = 0,1, \quad t > 0$$

$$u(0,x) = u_0(x), \quad 0 < x < 1.$$

More specifically, let

$$a(x) = \cosh(4x-2), \quad b(x) = \sinh(2x)$$

and choose u_0 , f , and g such that the exact solution becomes

$$u(t,x) = 1 - \exp\left(-\frac{x^2}{10t + 1/10}\right)$$

$$+ \frac{1}{2}(1+\tan(10t-11))(2\sin(\pi x) + 2\sin(2\pi t)\sin(2\pi x))$$

This oscillatory solution was selected in order to show the effectiveness of the approach. Figure 11 illustrates the form of the solution for various t .

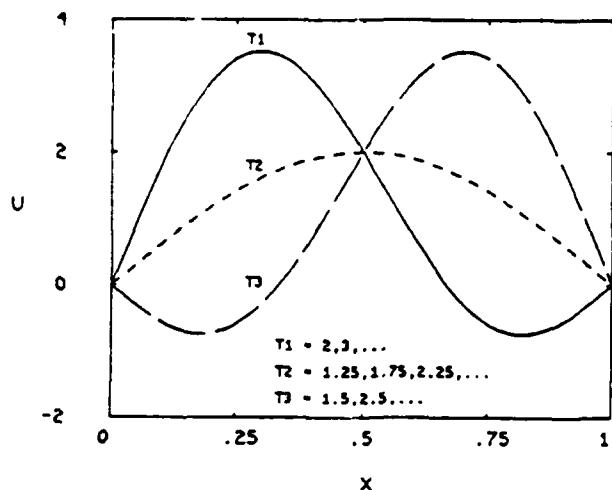


Figure 11

For the computation uniform meshes with N intervals and piecewise linear elements were used and the ODE-solver LSODI was applied. Figure 12 shows the relative errors, measured in the energy norm

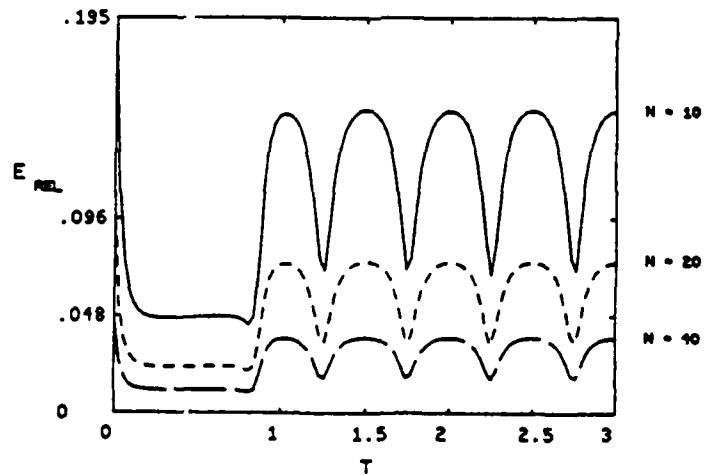


Figure 12

$$\|u(t, x)\|_E = \left[\int_0^1 (a(\frac{\partial u}{\partial x})^2 + bu^2) dx \right]^{\frac{1}{2}},$$

and Figure 13 the corresponding effectivity indices as functions of t and N .

Once again we see the high quality of the estimators even for relatively modest accuracies.

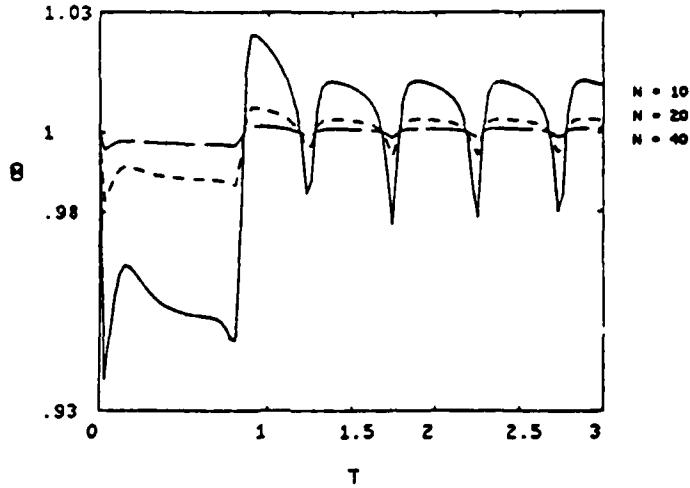


Figure 13

The interaction between the device of tolerance for the ODE-solver and the space discretization, as well as, the design of a total error estimator is discussed in [38]. The results can be extended to problems in higher space dimension and also to mildly nonlinear parabolic equations.

8. Nonlinear Problems

In recent years the application of finite element techniques to the numerical solution of nonlinear problems has been receiving increasing attention. In particular, the nonlinear analysis of the behavior of stationary mechanical systems has become more and more important in engineering applications. We refer, for example, see [1]. The area is rather large and we can present here only a brief overview of some questions and results relating to the general concepts discussed earlier.

8.1 Parametrical Systems: When we speak of the behavior of a mechanical structure, then we mean, for instance, the deformation of the structure in response to the action of external influence quantities such as load distributions, changes in material properties and geometrical data, etc. This implies that the equations which govern the deformation always involve a set of influence parameters, and, in line with the nonlinear character of these equations, we cannot, in general, draw conclusions about solutions for arbitrary parameter values from one such solution.

Hence, after suitable finite element discretizations we are led to consider finite-dimensional nonlinear equations of the form

$$F(y, p) = 0$$

where $y \in R^n$ is a vector of state variables (e.g., deformations), $p \in R^m$ a vector of parameters, and F a given function which maps $R^n \times R^m$ into R^n . The study of the response of the corresponding mechanical system now requires a determination of parts of the solution set

$$E(F) = \{(y, p) \in R^n \times R^m; F(y, p) = 0\}$$

and a study of its shape and features. In structural analysis $E(F)$ is often called the equilibrium surface of F .

Suppose now that F is of class C^r , $r \geq 1$, and that the derivative of F at $(y, p) \in R^n \times R^m$ is denoted by $DF(y, p)$. If the regularity set

$$R(F) = \{(y, p) \in R^n \times R^m; \text{rank } DF(y, p) = n\}$$

is non-empty, then the regular part,

$$E_R(F) = E(F) \cap R(F),$$

of the equilibrium surface of F is an open, m -dimensional C^r -manifold in $R^n \times R^m$. At a singular point $(y, p) \in E(F) - E_R(F)$ the surface may show a variety of different features but we will not discuss this here.

The characteristics of the equilibrium surface depend strongly on the selection of the parameters. Improperly selected parameters may well introduce meaningless singularities. Thus the selection of physically meaningful parameters, especially in neighborhoods of singular points is of considerable importance.

From a computational point of view it is not unexpected that the larger the set of free parameters the more involved is the task of analyzing the equilibrium surface. Hence, in practice, the standard approach is to restrict the variation of the parameter vector p to one degree of freedom. In other words, we require that p is a function $p = p(t)$ of one real variable t . Then the given equation reduces to an equation

$$\hat{F}(y, t) = F(y, p(t)) = 0, y \in R^n, t \in R^1$$

involving a single, real parameter t . If the function $p = p(t)$ is again of

class C^r , then the regular part $E_{\hat{R}}(\hat{F})$ of the equilibrium surface of \hat{F} constitutes a one-dimensional C^r -manifold, that is, a curve on $E(F)$.

It can be seen that we have $(y, t) \in R(\hat{F})$ exactly if one of the following two conditions holds:

(i) $D_y F(y, p(t))$ is non-singular.

(ii) rank $D_y F(y, p(t)) = n-1$ and $D_p F(y, p(t))p'(t) \notin \text{rge}(D_y F(y, p(t)))$

In either case we have $(y, r(t)) \in R(F)$. The points $(y, t) \in E_{\hat{R}}(\hat{F})$ where (ii) holds are called limit points or turning points of \hat{F} with respect to t .

On its regularity set $R(\hat{F})$ the function \hat{F} defines a direction field. More specifically, there exists a unique mapping T from $R(\hat{F})$ into R^{n+1} such that

$$\hat{DF}(y, t)T(y, t) = 0, \quad ||T(y, t)||_2 = 1, \quad \det \begin{pmatrix} \hat{DF}(y, t) \\ T(y, t)^T \end{pmatrix} > 0,$$

(see eq. [39]). Then (y, t) is a limit point of \hat{F} with respect to t exactly if the $(n+1)$ st component of $T(y, t)$ is zero.

In the case of structural problems, the critical points encountered on the solution curve defined by \hat{F} , that is, the limit points of \hat{F} or the points not in $R(\hat{F})$, signify a possible loss of stability of the structure. Such a loss of stability actually corresponds to a dynamic phenomenon whereby the structure undergoes a sudden change of deformation. The dynamics of this phenomenon are not specified by the equations involving F or \hat{F} , but it is possible to deduce from the shape of the equilibrium surface what type of sudden changes may be expected. Broadly speaking, at limit points we expect to encounter a snap-through or collapse behavior while at points not in $R(\hat{F})$ the structure may undergo (stable or unstable) buckling.

These observations already indicate that for a deeper understanding of the behavior of a structure it is necessary to analyze the form and principal features of the equilibrium surface of F . For this we need to consider continuation methods in a broader sense as a collection of numerical procedures for accomplishing tasks of the following type

- (i) Follow numerically any curve on the surface specified by a given combination $p = p(t)$ of parameter values.
- (ii) Determine on such a curve the critical points when stability may be lost.
- (iii) From any critical point follow a path in the boundary of the set of stable solutions on the surface.
- (iv) On any curve specified by (i) determine the location of points not in $R(F)$ and, in particular, of bifurcation points from which other paths emanate, and trace any such path.

Methods of these and related tasks have been developed by many authors in recent years (see eq. [40], [41], [42], [43], [44], [45], [46], [47], [48]). We shall not go into details here but sketch in the next section only some of the principal features of methods for the first two tasks (i) and (ii).

The discussion in this section certainly shows that in the nonlinear case the problem formulation as well as the computational requirements differ strongly from those in the linear case. In addition, there is now not only a need for estimates of the error of the computed solution but also of the various computed features of the equilibrium surface, as for instance, of the location of limit points or bifurcation points, etc. Furthermore, there is a requirement for a combined adaptive control of the space discretization and the steps of the continuation process. In a sense this is similar in concept to the situation discussed earlier in the case of parabolic equations.

8.2 A Basic Continuation Approach

As we saw our interest centers on computing continuous paths

$$x = x(s) = (y(s), t(s)) \in \mathbb{R}^{n+1}, s \in I$$

on $E_R(\hat{F})$ where \hat{F} is a reduced mapping of the form discussed in the previous section, and s is a suitable, as yet unspecified parameter ranging over some interval $I \in \mathbb{R}^1$. For ease of notation we denote the vectors $(y, t) \in \mathbb{R}^n \times \mathbb{R}^1$ simply by x .

For any $x \in R(\hat{F})$ the direction vector Tx defined earlier may be computed as follows

$$\begin{pmatrix} \hat{DF}(x) \\ (e^i)^T \end{pmatrix} v = e^{n+1}, Tx = \sigma \frac{v}{\|v\|_2}, \sigma = \frac{\text{sign det} \begin{pmatrix} \hat{DF}(x) \\ (e^i)^T \end{pmatrix}}{\text{sign}(e^i)^T v}$$

where e^1, \dots, e^{n+1} are the natural basis vectors of \mathbb{R}^{n+1} and e^i was chosen such that Tx has a component in its direction. If $x^k \in R(\hat{F})$ is a known approximation of our derived path $x = x(s)$ then

$$\pi(h) = x^k + h Tx^k, h \in \mathbb{R}^1$$

represents an approximation of the tangent line of the curve at x^k . Hence, if e^i again is such that $(e^i)^T Tx^k \neq 0$ then, for sufficiently small h , the hyperplanes $(e^i)^T(x - \pi(h)) = 0$ are transversal to the line $\pi = \pi(h)$ and hence also to the curve $x = x(s)$ itself. In other words, the augmented equation

$$g_h(x) = \begin{pmatrix} \hat{F}x \\ (e^i)^T(x - \pi(h)) \end{pmatrix} = 0$$

has a unique solution for any fixed, small h which may be computed, say, by

Newton's method applied to this equation with $\pi(h)$ as starting point.

With these observations we may outline a so-called locally parametrized continuation process as follows:

1. Given $x^0 \in E_R(\hat{F})$ and an initial index i_{-1} , set $k = 0$;
2. Compute Tx^k using the index $i = i_{k-1}$;
3. Choose $i = i_k$, $1 \leq i_k \leq n+1$, such that $(e^{i_k T})^T Tx^k \neq 0$;
4. Choose a steplength h along the line $\tau = \pi(h)$;
5. With $\pi(h)$ as starting point apply Newton's method to $g_h(x) = 0$;
6. If "convergence" then $x^{k+1} =$ computed solution, $k = k+1$, go to 2;
else go to 5 with $h = \frac{1}{2}h$.

In line with this schematic outline complete programs have been developed. For a specific version we refer to [49].

Because of the significance of limit points in many applications, it is not surprising that a number of algorithms have been proposed for the computation of such points. We mention here the methods presented in [49], [50], [51], [52], [53], [54], [55], [56], [57]. There are two basic classes of algorithms for computing limit points, namely (a) local iterative processes, and (b) methods utilizing the continuation path. The members of the first class may be started anywhere in the neighborhood of the desired limit point x^* while those in the second class require an input some point (or points) on the continuation path through x^* . In practice, this distinction tends to loose its significance since also in the first case a continuation process is needed to reach a neighborhood of x^* .

In [58] experimental comparisons of all the mentioned methods for a set of test problems were presented and some guidelines for their applicability given. It turns out that when two approximations bracketing the limit point x^* are available on the continuation path through x^* then a version of the method given in [49], [54] is highly reliable and efficient and hence can be recommended.

8.3 A Model Problem

Not unexpectedly, for nonlinear problems the theory of a posteriori error estimators has not yet advanced as far as in the linear case. However, a number of results have been developed and proved already and for a detailed survey we refer to [39]. Here we cannot enter into details, but restrict the discussion to some computational results for a specific model problem.

The problem under consideration concerns the planar deformations of a non-linearly elastic rod. More specifically, large deformations as well as material non-linearities are allowed. A comprehensive treatment of the global behavior of such rods was given in [59]. These rods can suffer not only flexure but also compression (or extension) and shear. The basic configuration of the rod is sketched in Figure 14.

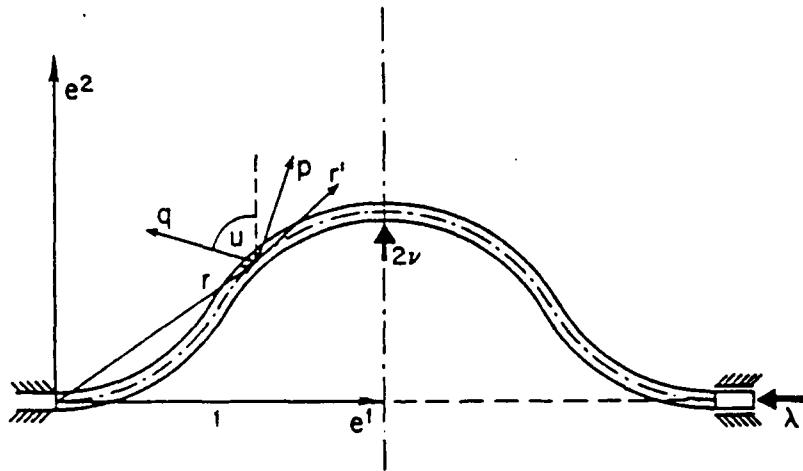


Figure 14

We restrict ourselves to the case of a symmetric shape where it suffices to consider only one half of the rod, say the left half in the figure. Then the configuration of the rod is described by two functions

$$r: [0,1] \rightarrow \mathbb{R}^2, \quad q: [0,1] \rightarrow \mathbb{R}^2, \quad \|q(s)\|_2 = 1$$

where r defines the axis of the deformed rod and q the direction of the cross-section. Let e^1, e^2 be the global basis vectors as shown in the figure and u the angle between q and e^2 . Under the influence of a constant load with the components λ and v , as indicated in the figure, the problem can then be modeled by the two-point boundary-value problem

$$\begin{aligned} \frac{d}{ds} a(u', u, \lambda, v) + (\lambda \sin u + v \cos u)(1 + b(-\lambda \cos u + v \sin u)) \\ + (\lambda \cos u - v \sin u)c(\lambda \sin u + v \cos u) = 0, \quad 0 < s < 1, \\ u(0) = u(1) = 0, \end{aligned}$$

with given functions a , b , c representing constitutive equations.

For our computations we employed the standard finite element method with C^0 elements of degree p .

For the specific choice of the coefficient functions

$$a(t) = c(t) = \arctan \frac{1}{2}t, \quad b(t) = \begin{cases} \frac{t}{1-t} & \text{for } t < 0 \\ t+2t^2 & \text{for } t \geq 0 \end{cases}$$

Figure 15 shows several solution curves for $v = \text{constant}$ plotted in a coordinate system with λ as abscissa and the norm $\|\tilde{u}\|_{L^\infty}$ of the computed solution as ordinate. In all cases uniform meshes with eight sub-intervals and cubic C^0 -elements were used.

In order to assess the effectivity of the error estimation, we used a "nearly exact" solution the computed solution with a uniform mesh of 10 subintervals and quintic C^0 -elements. We consider the curve $v = 0.1$ and on it the five target points $\lambda = 1.0, 2.5, 3.5, 2.5, 1.6$. Table 7 shows the computed error estimators of the error in the H^1 norm and the H^1 -norm of the "nearly exact" error in percent of the norm of the solution. The table shows clearly the high quality of the estimators. In the cases left blank in the table the error is essentially zero within machine accuracy.

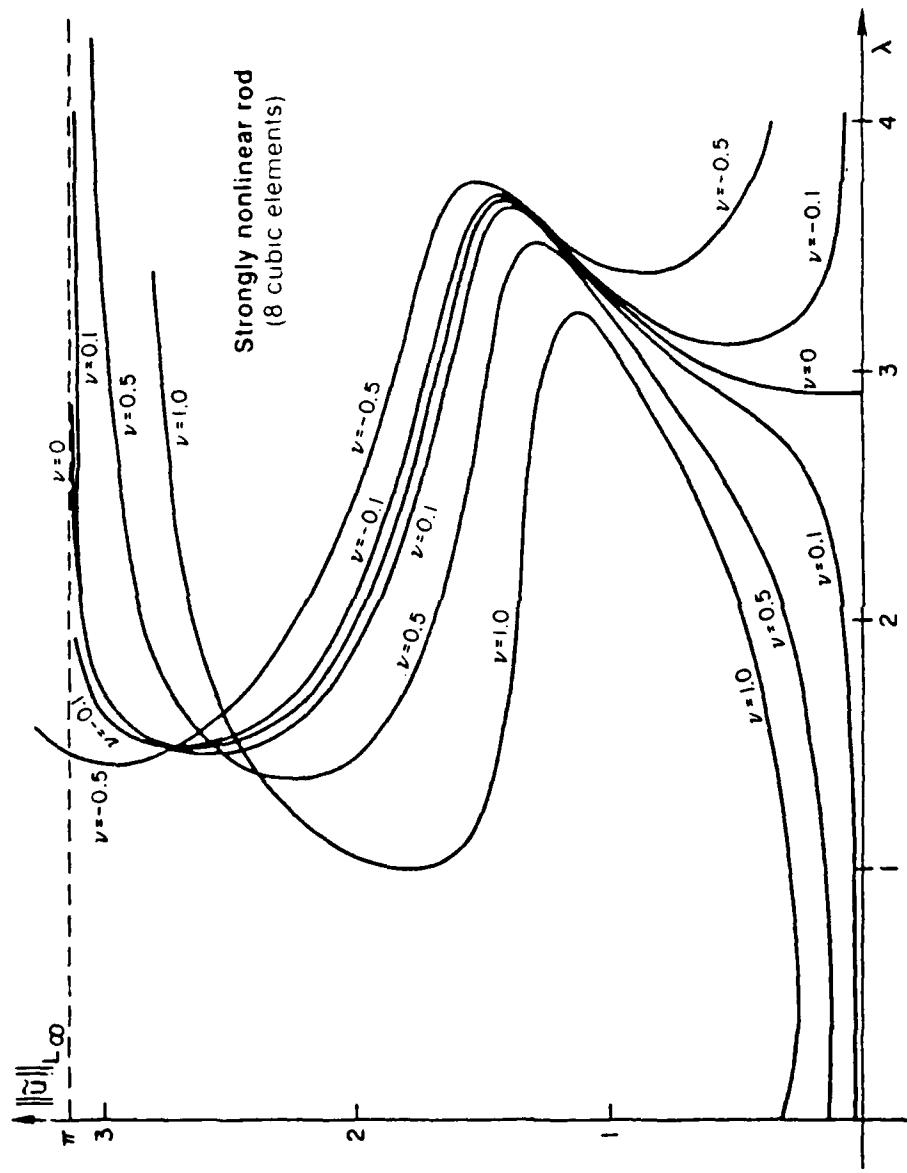


Figure 15

	ρ	1	2	3	4
Error estim.	1.0	12.15%	0.1137%	0.004089%	
Error norm		12.13%	0.1138%	0.004089%	
Error estim.	2.5	11.55%	0.4306%	0.01552%	
Error norm		12.04%	0.4310%	0.01552%	
Error estim.	3.5	21.58%	1.597%	1.123%	0.5011%
Error norm		23.91%	1.900%	1.391%	0.5389%
Error estim.	2.5	21.53%	1.708%	1.003%	0.5013%
Error norm		21.88%	1.865%	1.285%	0.5455%
Error estim.	1.6	19.45%	0.7632%	0.8123%	0.3171%
Error norm		19.14%	1.135%	0.9576%	0.3312%

Table 8

As Figure 15 shows there are two limit points along the curve for $\nu = 0.1$. With a mesh of eight equal, cubic C^0 -elements these limit points occur at $\lambda_{T_1} \doteq 3.6546$ and $\lambda_{T_2} \doteq 1.46749$. Table 7 shows the effectivity of our error-estimators of these critical values. For the comparison five and ten equal, quadratic C^0 -elements were used. Again the estimators perform very well.

In general the error varies considerably along any one solution curve.

Limit Point	No. of Interv.	Est. of Solution Error	Limit Point		
			Value	Error	Estim.
λ_{T_1}	5	13.11%	3.699385	0.04472	0.05294
	10	3.18%	3.657982	0.00317	0.00399
λ_{T_2}	5	7.205%	1.470290	0.002793	0.002843
	10	1.613%	1.467897	0.00040	0.000357

Table 8

Thus adaptive approaches are extremely important in order to control this error along the curve. Our experimentation has shown that savings of about 70% or more in computational effort can be achieved for problems of the type discussed here. For higher dimensional problems such savings become extremely valuable. For more details we refer to [39].

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